ECE 346: Digital Signal Processing
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Spring 2021
Week of April 26, 2021

Recitation #11: Spectral Analysis & DFT Properties

Objective & Outline

- Problems $1-4$: recitation problems
- Problem 5: self-assessment problem

The problems start on the following page.

Problem 1 (Computing DFT)**.** Consider the discrete sequence

$$x[n] = \cos\left(\frac{\pi}{7}n\right), \quad n = 0, 1, \ldots, 55. \tag{1}$$

(a) Derive a closed-form expression for the 56-point DFT of $x[n]$.

(b) Provide a labeled stem plot for the magnitude of the 56-point DFT of $x[n]$, $|X[k]|$.

(c) Will the 56-point DFT of the sequence

$$\tilde{x}[n] = \cos\left(\frac{\pi}{7}n\right), \quad n = 0, 1, \ldots, 49. \tag{2}$$

be the same as the 56-point DFT of $x[n]$? Justify your answer.

Problem 1:-

(a) Recall the analysis equation for the N-point DFT:

$$X[k] = \sum_{n=0}^{N-1} x[n] W_N^{kn},$$

where $W_N = e^{-j\frac{2\pi}{N}}$.

We can "euler-ize" $x[n]$ and then explicitly take its DFT:

$$x[n] = \cos\left(\frac{\pi}{7}n\right)$$
$$= \frac{1}{2}\left[e^{j\frac{\pi}{7}n} + e^{-j\frac{\pi}{7}n}\right].$$

Using the linearity property of the DFT:

$$X[k] = \sum_{n=0}^{55} x[n] W_{56}^{kn}$$

$$= \frac{1}{2}\left[\sum_{n=0}^{55} e^{j\frac{\pi}{7}n} W_{56}^{kn} + \sum_{n=0}^{55} e^{-j\frac{\pi}{7}n} W_{56}^{kn}\right]$$

Note that

$$e^{j\frac{\pi}{7}n} = e^{j\frac{8\pi}{56}n} = e^{j\frac{2\pi}{56}\cdot 4n}$$
$$= W_{56}^{-4n}.$$

$$e^{-j\frac{\pi}{7}n} = W_{56}^{4n}$$
$$= W_{56}^{-52n}$$

Plugging this in,

$$X[k] = \frac{1}{2}\left[\sum_{n=0}^{55} W_{56}^{(k-4)n} + \sum_{n=0}^{55} W_{56}^{(k-52)n}\right]$$
$$= \frac{1}{2}\left[\underbrace{\frac{1 - W_{56}^{(k-4)56}}{1 - W_{56}^{(k-4)}}}_{=0 \ \forall k \neq 4} + \underbrace{\frac{1 - W_{56}^{(k-52)56}}{1 - W_{56}^{(k-52)}}}_{=0 \ \forall k \neq 52}\right].$$

Note that the first term is 0 for all $k \neq 4$ by doing L'Hopital's rule:

$$\left.\frac{56\, W_{56}^{(k-4)56}}{W_{56}^{(k-4)}}\right|_{k=4} = 56.$$

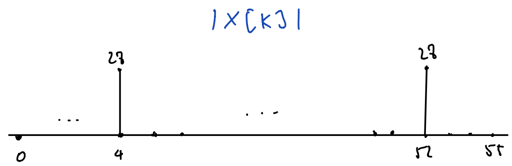
This also applies for the second term. Thus,

$$X[k] = 28\,\delta[k-4] + 28\,\delta[k-52].$$

(b) Plotting

$$X[k] = 28\,\delta[k-4] + 28\,\delta[k-52],$$

we get



(c) Clearly, the resulting DFT would not be the same, since the numerators would be

$$1 - W_{50}^{(k-4)50} \quad \text{and} \quad 1 - W_{50}^{(k-52)50},$$

where then $k = 4, 52$ would yield $X[k] = 25, 25$, where $25 \neq 28$.

Problem 2 (Spectral Analysis)**.** Consider a continuous-time signal $x(t)$ that is bandlimited to 5kHz. We are interested in carrying out DFT-based spectral analysis of the signal $x(t)$.

(a) What should be the constraint on the sampling period T?

(b) Assuming that the sampling period is $T = \frac{1}{20{,}000}$ seconds, how many DFT points must we take to ensure that the continuous-time frequencies are spaced no farther than 5Hz apart?

(c) Suppose we wanted our signal to be windowed to $5,000$ samples to reduce spectral leakage. What is the new spectral resolution, assuming the same sampling period and the minimum number of DFT points?

Problem 2 :-

(a) Recall that $x(t)$ is bandlimited to 5kHz. To satisfy Nyquist criteria, f_s should satisfy

$$f_s \geq 2(f_m) = 2(5\text{kHz})$$

$$\therefore f_s \geq 10\text{kHz}.$$

Thus, the sampling period should satisfy

$$T \leq \frac{1}{10000} \text{ seconds}.$$

(b) Recall that the spectral resolution in Hz is

$$\text{Resolution} = \frac{1}{NT}.$$

Plugging in $T = 1/20000$,

$$\frac{20000}{N} \leq 5$$

$$\Rightarrow 4000 \leq N.$$

The DFT points needed should satisfy

$$N \geq 4000 \text{ samples}.$$

(c) Using the same equation,

$$\text{Resolution} = \frac{1}{NT} = \frac{20000}{5000} = 4 \text{ Hz}.$$

Thus, the new spectral resolution is 4 Hz.

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Problem 3 (More Spectral Analysis)**.** Consider a continuous-time signal $x(t)$ whose continous-time Fourier transform (CTFT) is given by

$$X(j\Omega) = \begin{cases} |\Omega| e^{-j\Omega}, & |\Omega| \leq 100\pi \\ 0, & \text{otherwise.} \end{cases} \tag{3}$$

It is desired to carry out spectral analysis of $x(t)$ using the N-point DFT $X[k]$ of its samples $x[n] = n(nT)$.

(a) Provide two unique choices of the pair (N, T) that will lead to sampling of the CTFT of $x(t)$ in which the samples are exactly 5Hz apart.

(b) Assuming $T = 1/200$ and $N = 40$ and ignoring the effects of spectral leakage, derive an expression for the DFT $X[k]$ for $k = 0, 1, \ldots, 39$.

Problem 3 :-

(a) Note that in order to satisfy Nyquist criteria,

$$f_s \geq 2(50) = 100 \text{ Hz},$$

the sampling period must satisfy $T \leq \frac{1}{100}$ seconds. Choosing $T = 1/100$ and $1/200$, we get the following choices of N:

① $5 = \frac{100}{N} \Rightarrow N = 20$ ② $5 = \frac{200}{N} \Rightarrow N = 40$.

Thus, two pairs (N, T) would be

$$\left\{ \left(20, \tfrac{1}{100}\right), \left(40, \tfrac{1}{200}\right) \right\}.$$

(b) Recall that

$$X[k] = \frac{1}{T} X(j\Omega)\Big|_{\Omega = \frac{2\pi k}{NT}} \quad \text{for } k = 0, \dots, N/2$$

$$X[k] = \frac{1}{T} X(j\Omega)\Big|_{\Omega = \frac{2\pi(k-N)}{NT}} \quad \text{for } k = N/2 + 1, \dots, N-1.$$

Note that

$$X(j\Omega) = 0 \quad \forall\, \Omega > 100\pi,$$

and so

$$X[k] = 0 \quad \text{for } \frac{2\pi k}{NT} > 100\pi,$$

which equates to

$$\frac{2\pi k}{NT} > 100\pi = \frac{2\pi k \cdot 200}{40} > 100\pi$$

$$= 10\pi k > 100\pi$$

$$\Rightarrow k > 10.$$

Since the positive frequencies relate to $k = 0, \dots, 20$, $X[k]$ is then 0 from

$$10 < k \leq 20.$$

Similarly,

$$X(j\Omega) = 0 \quad \forall k < -100\pi.$$

This translates into

$$X[k] = 0 \quad \text{for } 21 \leq k < 30.$$

Thus,

$$X[k] = 0 \quad , \; k = 11, \dots, 29$$

$$X[k] = 200 \cdot \frac{2\pi k}{NT} \cdot e^{-j \frac{2\pi k}{NT}}$$

$$= 2000\pi k e^{-j 10\pi k} \quad , \quad k = 0, \cdots, 10 \cdot$$

$$X[k] = 200 \cdot \frac{2\pi |k-N|}{NT} \cdot e^{-j \frac{2\pi (k-N)}{NT}}$$

$$= 2000\pi (40-k) e^{-j 10\pi (k-40)} \quad , \quad k = 30, \dots, 39.$$

Writing this out, we get

$$X[k] = \begin{cases} 2000\pi k e^{-j 10\pi k} & , \quad k = 0, 1, \dots, 10, \\ 0 & , \quad k = 11, 12, \dots, 29, \\ 2000\pi (40-k) e^{-j 10\pi (k-40)} & , \quad k = 30, 31, \dots, 39. \end{cases}$$

Problem 4 (Properties of the DFT)**.** Consider the discrete-time signal $x[n]$ defined as

$$x[n] = \delta[n-1] - \delta[n-3] + 3\delta[n-4]. \tag{4}$$

Provide labeled plots of the following sequences.

(a) $x[\langle n-3 \rangle_5]$

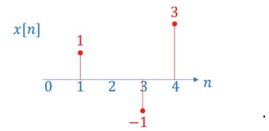
(b) $x[\langle n-3 \rangle_8]$

(c) $x[\langle n+2 \rangle_5]$

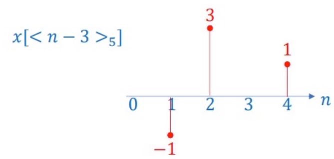
(d) $x[\langle -n \rangle_5]$

Problem 4 :-

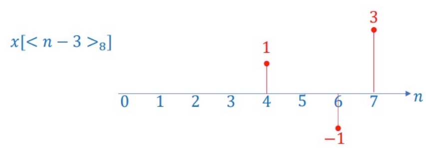
Note that the sequence of x[n] originally looks like



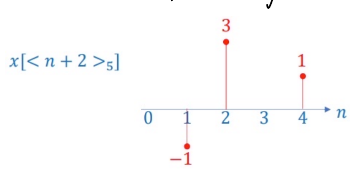
(a) Performing a circular shift of 3, we get



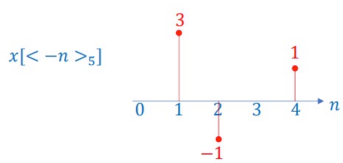
(b) Performing a circular shift of 3, we get



(c) Performing a circular shift of -2, we get



(d) Lastly, a circular flip yields the plot



Problem 5 (Self-assessment). Suppose that the continuous-time signal $x(t)$ was sampled with a sampling frequency of 10kHz. Spectral analysis was performed on the resulting length-10 discrete-time sequence $x[n]$. The 10-point DFT of $x[n]$ is given by

$$X[k] = 3\delta[k-1] - 2e^{j\frac{\pi}{4}}\delta[k-4] + \delta[k-7] + 4e^{j\frac{\pi}{2}}\delta[k-9]. \tag{5}$$

Ignoring spectral leakage, provide solutions for the following:

(a) What is the spectral resolution of this system?

(b) What continuous-time frequencies are present in this signal according to the 10-point DFT?

Problem 5 :-

(a) Given $T = 1/10000$ seconds and $N = 10$, the spectral resolution is

$$\text{Resolution} = \frac{1}{NT}$$
$$= 1 \text{ kHz}.$$

(b) Firstly, note that $X[k]$ is defined for $k = 0, 1, \ldots, 9$, and it is non-zero for $k = 1, 4, 7, 9$. Also, $k = 1$ and $k = 4$ relate to the positive frequencies of $X(j\Omega)$, while $k = 7$ and $k = 9$ yield the negative frequencies. Thus, using

$$\Omega = \frac{2\pi k}{NT}$$

and

$$\Omega = \frac{2\pi k}{NT} - \frac{2\pi}{T},$$

the continuous-time frequencies present are

$$k = 1 \rightarrow \Omega = 2000\pi$$
$$k = 4 \rightarrow \Omega = 8000\pi$$
$$k = 7 \rightarrow \Omega = -6000\pi$$
$$k = 9 \rightarrow \Omega = -2000\pi.$$

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